Introduction to Lattice QCD (I)

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Outline

- Introduction to QCD
- Basic Notions of Lattice Field Theory
- Gauge Field on the Lattice
- Naive Lattice Fermion
- Wilson-Dirac Operator
Quantum Chromodynamics (QCD) is the quantum field theory for the strong interaction between quarks and gluons.

Salient features:

1. Gauge group $SU(3) \Rightarrow$ gluons have self-interactions.
2. Asymptotic freedom: $g(r) \to 0$ as $r \to 0$.
3. IR slavery: $g(r) \approx 1$ as $r \approx 1$ fm $\Rightarrow$ quark (color) confinement.
4. No exact analytic solutions (similar to any 4d QFT)
Quarks

Quarks are spin $\frac{1}{2}$ fermions carrying color, and there are 6 species (flavors) of quarks.

\begin{align*}
\text{u} & \text{c} \text{t} & \text{u} & \text{c} & \text{t} & \text{u} & \text{c} & \text{t} \\
\text{d} & \text{s} & \text{b} & \text{d} & \text{s} & \text{b} & \text{d} & \text{s} & \text{b}
\end{align*}

Hadrons are color singlets of quarks.

\begin{align*}
P &= uud + \text{antisym. in color} \\
N &= udd + \text{antisym. in color} \\
\pi^+ &= u\bar{d} + u\bar{d} + u\bar{d}
\end{align*}

The nuclear force between nucleons emerges as residual interactions of QCD.
The action of QCD

\[ S_{QCD} = \int d^4x \left\{ \sum_{\text{flavors}} \bar{\psi}_f \left[ i\gamma_\mu (\partial_\mu + igA_\mu) - m_f \right] \psi_f - \frac{1}{2} \text{tr}(F_{\mu\nu}F_{\mu\nu}) \right\} \]

where

\[ A_\mu = T^a A^a_\mu, \quad \text{tr}(T^aT^b) = \delta^{ab}/2 \]
\[ F_{\mu\nu} = T^a F^a_{\mu\nu}, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu A^c_\nu \]

\( T^a, a = 1, \cdots, 8 \) are Hermitian 3 \times 3 matrices, the generators of \( SU(3) \) group.

The color and Dirac indices of quark fields are suppressed.

Explicitly, for \( u \) quark at \( x = (\vec{x}, t) \)

\[ \psi_f \leftrightarrow u_{c\alpha x}, \quad \alpha = 1, 2, 3, 4, \quad c = r, g, y \]
Meson Operators

Meson

\[(\bar{q}\Gamma Q)_x = \sum_{a,\alpha,\beta} \bar{q}_{a\alpha x} \Gamma_{\alpha\beta} Q_{a\beta x}\]

where

- \(q, Q\) – quark fields
- \(\bar{q}, \bar{Q}\) – antiquark fields
- \(x, y\) – lattice sites
- \(\alpha, \beta\) – Dirac indices
- \(a, b, c\) – color indices

\[\begin{align*}
(\pi^+)_x &= (\bar{d}\gamma^5 u)_x = \sum_{a,\alpha,\beta} \bar{d}_{a\alpha x} \gamma^5_{\alpha\beta} u_{a\beta x} \\
(K^0)_x &= (\bar{s}\gamma^5 d)_x = \sum_{a,\alpha,\beta} \bar{s}_{a\alpha x} \gamma^5_{\alpha\beta} d_{a\beta x}
\end{align*}\]
Baryon Operators

The basic components of any baryon operator are “diquarks"

\[
(q^T \Gamma Q)_{a,x} = \sum_{b,c,\alpha,\beta} \epsilon_{abc} q_{b\alpha x} \Gamma_{\alpha\beta} Q_{c\beta x}
\]

\[
[q^T \Gamma Q]_{a,x} = \sum_{b,c,\alpha,\beta} \epsilon_{abc} (q_{b\alpha x} \Gamma_{\alpha\beta} Q_{c\beta x} - Q_{b\alpha x} \Gamma_{\alpha\beta} q_{c\beta x})
\]

Scalar: \( \Gamma = C \gamma_5 \), \( C \) is the charge conjugation operator

Pseudoscalar: \( \Gamma = C \)

Vector: \( \Gamma = C \gamma^\mu \), \( \mu = 1, 2, 3 \)
Spin-1/2 Baryon Operators

\[ N = [u^T C \gamma_5 d]d \]
\[ \Xi = [u^T C \gamma_5 s]s \]
\[ \Sigma = [u^T C \gamma_5 s]u \]
\[ \Lambda = [d^T C \gamma_5 s]u + [s^T C \gamma_5 u]d - 2[u^T C \gamma_5 d]s \]
\[ \Lambda_1 = [d^T C \gamma_5 s]u + [s^T C \gamma_5 u]d + [u^T C \gamma_5 d]s \]
\[ \Lambda_c = [u^T C \gamma_5 d]c \]
\[ \Sigma_c = (d^T C \gamma_5 d)c \]
\[ \Omega_c = (s^T C \gamma_5 s)c \]
\[ \Xi_c = (u^T C \gamma_5 s)c \]
\[ \Xi_{cc} = [u^T C \gamma_5 c]c \]
\[ \Omega_{cc} = [s^T C \gamma_5 c]c \]
Consider the interpolating operator for $\Delta$,

$$\Delta_{x\alpha}^\mu = (u^T C \gamma^\mu u)_{xa} u_{x\alpha a}$$

which overlaps with $J = 1/2$ and $J = 3/2$ states. Thus spin projections are required.

$$C_{\mu\nu}^{3/2}(t) = \sum_{\sigma=1}^{3} \left( \delta_{\mu\sigma} - \frac{1}{3} \gamma_\mu \gamma_\sigma \right) C^{\sigma\nu}(t)$$

$$C_{\mu\nu}^{1/2}(t) = \frac{1}{3} \sum_{\sigma=1}^{3} \gamma_\mu \gamma_\sigma C^{\sigma\nu}(t), \quad C^{\sigma\nu}(t) = \sum_{\vec{x}} \langle \Delta^\sigma (\vec{x}, t) \bar{\Delta}^{\nu}(\vec{0}, 0) \rangle$$

Then the masses of $J = 3/2^\pm$ states can be extracted from any one of the 9 possibilities ($\mu, \nu = 1, 2, 3$) of $C_{\mu\nu}^{3/2}(t)$. 
\[ \Delta^\mu = (u^T C \gamma^\mu u) u \]
\[ \Sigma^\mu = (u^T C \gamma^\mu u)s + (s^T C \gamma^\mu u)u + (u^T C \gamma^\mu s)u \]
\[ \Xi^\mu = (u^T C \gamma^\mu s)s + (s^T C \gamma^\mu u)s + (s^T C \gamma^\mu s)u \]
\[ \Lambda^\mu_c = (u^T C \gamma^\mu d) c \]
\[ \Sigma^\mu_c = (u^T C \gamma^\mu u)c \]
\[ \Xi^\mu_c = (u^T C \gamma^\mu s)c \]
\[ \Omega^\mu_c = (s^T C \gamma^\mu s)c \]
\[ \Xi^\mu_{cc} = (u^T C \gamma^\mu c)c \]
\[ \Omega^\mu_{cc} = (s^T C \gamma^\mu c)c \]
\[ \Omega^\mu_{ccc} = (c^T C \gamma^\mu c)c \]
The Challenges of QCD

At the hadronic scale, $g(r) \sim 1$, perturbation theory is incapable to extract any quantities from QCD, nor to tackle the most interesting physics, namely, the spontaneously chiral sym. breaking and the color confinement.

To extract any physical quantities from the first principles of QCD, one has to solve QCD nonperturbatively.

A viable nonperturbative formulation of QCD was first proposed by Wilson in 1974.

But, to solve the problem of lattice fermion, and to formulate exact chiral symmetry on the lattice had not been resolved until 1992-98.
1. Perform Wick rotation: $t \rightarrow -ix_4$, then $\exp(iS) \rightarrow \exp(-S_E)$, and the expectation value of any observable $O$

$$\langle O \rangle = \frac{1}{Z} \int [dA][d\psi][d\bar{\psi}] \ O(A, \psi, \bar{\psi}) \ e^{-S_E}$$

$$Z = \int [dA][d\psi][d\bar{\psi}] \ e^{-S_E}$$

(Recall the divergences in QFT, which requires reg. and ren., stemming from $\infty$ d.o.f., and proximity of any field ops.)

2. Discretize the space-time as a 4-d lattice $L^4 = (Na)^4$ with lattice spacing $a$. Then the path integral in QFT becomes a well-defined multiple integral which can be evaluated numerically via Monte Carlo

$$\langle O \rangle = \frac{1}{Z} \int \prod_i dA_i \prod_j d\psi_j \prod_k d\bar{\psi}_k \ O(A, \psi, \bar{\psi}) \ e^{-S_E}$$
Gluon Fields on the lattice

The $SU(3)$ color gluon field $A_\mu(x)$ are defined on each link connecting $x$ and $x + a\hat{\mu}$, through the link variable

$$U_\mu(x) = \exp \left[ i a g a_\mu \left( x + \frac{a}{2}\hat{\mu} \right) \right]$$

Then the gluon action on the lattice can be written as

$$S_g[U] = \frac{6}{g^2} \sum_{\text{plaquette}} \left[ 1 - \frac{1}{3} \text{Re} \ tr(U_p) \right] \xrightarrow{a \to 0} \int d^4x \, \frac{1}{2} \text{tr}[F_{\mu\nu}(x)F_{\mu\nu}(x)]$$

where $U_p = U_\mu(x)U_\nu(x + a\hat{\mu})U^\dagger_\mu(x + a\hat{\nu})U^\dagger_\nu(x)$
The fermion fields $\bar{\psi}(x)$ and $\psi(x)$ are defined at each site $x = (n_1, n_2, n_3, n_4)a$, $1 \leq n_i \leq N$.

But, to discretize $D = \gamma_\mu (\partial_\mu + iA_\mu)$ turns out to be a very difficult problem!

It is impossible to preserve the chiral symmetry

$$D\gamma_5 + \gamma_5 D = 0$$

on the 4d lattice without violating at least one of the basic properties of Dirac fermion field.
Naive Lattice Fermion

\[ \partial_\mu \psi(x) \leftrightarrow \frac{1}{2a} [\psi(x + a\hat{\mu}) - \psi(x - a\hat{\mu})] \]

The naive lattice free fermion action is

\[ S_{naive} = \sum_{\mu, x, y} \overline{\psi}(x) \gamma_\mu \left( \frac{\delta_{x+a\hat{\mu}, y} - \delta_{x-a\hat{\mu}, y}}{2a} \right) \psi(y) \]

Its propagator is

\[ D^{-1}(x, y) \simeq \int_{-\pi/a}^{\pi/a} d^4p \ e^{ip \cdot (x-y)} \frac{a}{i\gamma_\mu \sin(p_\mu a)} \]

which has poles at \( p_\mu = 0 \) and \( p_\mu a = \pi \).

\[ p_\mu \simeq 0, \quad \sin(p_\mu a)/a \simeq p_\mu \]

\[ p_\mu \simeq \pi/a - p'_\mu, \quad \sin(p_\mu a)/a \simeq -p'_\mu \]
Consequently, we end up with $2^4 = 16 = 1 + 15$ fermion species rather than one. (Recall, in continuum, only one pole at $p = 0$.)

$$p_\mu \simeq 0, \quad \sin(p_\mu a)/a \simeq p_\mu$$

$$p_\mu \simeq \pi/a - p'_\mu, \quad \sin(p_\mu a)/a \simeq -p'_\mu$$

Since the "-" sign of $-p'_\mu$ can be absorbed by redefining the $\gamma_\mu$, and $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, the chirality of the 15 doubled modes are divided into two groups: 7 with $+\gamma_5$, and 8 with $-\gamma_5$.

In a gauge background, each fermion species contributes the same magnitude of axial anomaly to the divergence of axial vector current, but with the sign $\pm$ same as its sign of $\pm \gamma_5$. Thus the axial anomaly of naive lattice fermion must be zero.
The Wilson-Dirac Operator (1975)

\[
D_W = \sum_{\mu} \gamma_\mu t_\mu + W
\]

\[
t_\mu(x, y) = \frac{1}{2a} \left[ \delta_{x+a\hat{\mu}, y} - \delta_{x-a\hat{\mu}, y} \right]
\]

\[
W(x, y) = \frac{1}{2a} \sum_{\mu} \left[ 2\delta_{x,y} - \delta_{x+a\hat{\mu}, y} - \delta_{x-a\hat{\mu}, y} \right]
\]

The Wilson fermion propagator is

\[
D^{-1}(x, y) \sim \int_{-\pi/a}^{\pi/a} d^4p \ e^{ip\cdot(x-y)} \frac{a}{\sum_\mu [i\gamma_\mu \sin(p_\mu a) + 1 - \cos(p_\mu a)]}
\]
The purpose of the Wilson term $W$ is to give each doubled mode a mass $\sim 1/a$ such that in the continuum limit ($a \to 0$), each doubled mode becomes infinitely heavy and decouples from the fermion propagator. However, it breaks chiral symmetry explicitly and gives $O(a)$ lattice artifacts.

But the chiral symmetry plays an important role in QCD! It forbids an additive quark mass renormalization, and its spontaneously breaking provides (nearly) Goldstone bosons with their specific interactions.

Therefore, one needs to preserve the chiral symmetry on the lattice, otherwise, one hardly uses the lattice regularization to investigate the low energy phenomenology of QCD.